FP1 - January 2010

(1) a) $\alpha + \beta = 6/3 = 2$

$\alpha\beta = 1/3$

b) $\alpha^3 + \beta^3 = (\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)$

$= 2^3 - 3 \times 1/3 \times 2 \quad = 6$

c) $\boxed{\text{SUM}}$ $\quad \alpha^2/\beta + \beta^2/\alpha = \alpha^3/\alpha\beta + \beta^3/\alpha\beta$

$= \dfrac{\alpha^3 + \beta^3}{\alpha\beta} = 6/\tfrac{1}{3} = 18$

$\boxed{\text{PRODUCT}}$ $\quad \alpha^2/\beta \times \beta^2/\alpha = \alpha^2\beta^2/\alpha\beta = \alpha\beta = 1/3$

$\rightarrow \quad x^2 - \boxed{\text{SUM}}\,x + \boxed{\text{PRODUCT}} = 0$

$\rightarrow \quad x^2 - 18x + 1/3 = 0$

$\rightarrow \quad 3x^2 - 54x + 1 = 0$

(2) a) $z^2 = (1+i)(1+i) = 1 + 2i + i^2$

$= 1 + 2i - 1 = 2i$

b) $z^8 = (z^2)^4 = (2i)^4 = 2^4 \times i^4$

$= 16 \times (-1)(-1) = 16$

c) $(z^*)^2 = (1-i)(1-i) = 1 - 2i + i^2 = -2i$

$= -z^2$

(3) $\theta = 2n\pi + a, \quad \theta = 2n\pi + (\pi - a)$

Key angle: $\sin^{-1}(1) = \pi/2$

$4x + \pi/4 = 2n\pi + \pi/2, \quad 4x + \pi/4 = 2n\pi + \pi/2$

SAME!

$\rightarrow \quad 4x = 2n\pi + \pi/4$

$\rightarrow \quad x = \frac{1}{2}n\pi + \pi/16$

(4) a) $A - I = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}$

$(A - I)^2 =$

$$\begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix} = 12I$$

b) $A - B = \begin{bmatrix} 1 & 4 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ p & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 3-p & 0 \end{bmatrix}$

$(A - B)^2$

$$\begin{bmatrix} 0 & 1 \\ 3-p & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3-p & 0 \end{bmatrix} = \begin{bmatrix} 3-p & 0 \\ 0 & 3-p \end{bmatrix}$$

$3 - p = 12$

$\rightarrow p = -9$

5) a) The integral is undefined when $x = 0$

As $x \to 0$, $1/\sqrt{x} \to \infty$

b) i) $\int_0^{1/16} x^{-1/2} = \int_p^{1/16} x^{-1/2}$

$= \left[2x^{1/2} \right]_p^{1/16} = \left[2(1/16)^{1/2} - 2p^{1/2} \right]$

$= 1/2 - 2\sqrt{p}$

As $p \to 0$, $2\sqrt{p} \to 0$, so $\int \to 1/2$

ii) $\int_0^{1/16} x^{-5/4} = \int_p^{1/16} x^{-5/4}$

$= \left[-4x^{-1/4} \right]_p^{1/16} = \left[-4/\sqrt[4]{x} \right]_p^{1/16}$

$= -4/\sqrt[4]{1/16} - -4/\sqrt[4]{p}$

As $p \to 0$, $4/\sqrt[4]{p} \to \infty$, so integral has no finite value.

6) a) i)

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 9 & 9 & 3 \\ 2 & 2 & 4 & 4 \end{bmatrix} \rightarrow (3,2) \quad (9,2) \quad (9,4) \quad (3,4)$$

ii) See Mark Scheme

b) i) See Mark Scheme.

ii) Rotation 90° CW $= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

c) $R_1 \to R_3$ must multiply in reverse order

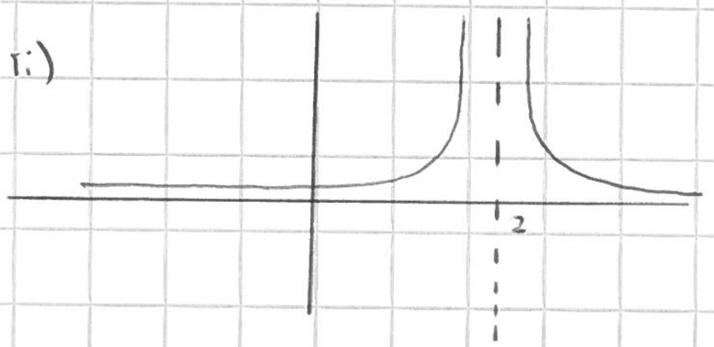
$\to \quad \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$

(7) a) i) $\boxed{x = 2}$ As $x \to \infty$, $y \to {}^{1}/_{\infty} = 0$

$\to \boxed{y = 0}$

ii)



b) $x - 3 = \dfrac{1}{(x-2)^2}$

$\to (x-3)(x-2)^2 = 1$

$\to (x-3)(x-2)^2 - 1 = 0$

$f(x) = (x-3)(x-2)^2 - 1$

$f(3) = -1$

$f(4) = 3$

Sign change, $\therefore$ root lies between 3 & 4 ii)

Interval	Midpoint (x)	Value of $f(x)$
3 - 4	3.5	0.125 ~~-0.0055555...~~
3 - 3.5	3.25	-0.984375

→ Between 3.25 & 3.5

∴ must be between 3.25 & 3.5

(8) a) $\sum r^3 + \sum r$

$= \frac{1}{4} n^2 (n+1)^2 + \frac{1}{2} n(n+1)$

$= \frac{1}{4} n(n+1) + [n(n+1) + 2]$

$= \frac{1}{4} n(n+1) + [n^2 + n + 2]$

b) $8 \sum r^2 = \frac{8}{6} n(n+1)(2n+1)$

$\rightarrow \frac{1}{4} n(n+1)(n^2+n+2) = \frac{8}{6} n(n+1)(2n+1)$

$\div n(n+1)$ $\rightarrow \frac{1}{4}(n^2+n+2) = \frac{8}{6}(2n+1)$

$\times 12$ $\rightarrow 3(n^2+n+2) = 16(2n+1)$

$\rightarrow 3n^2 + 3n + 6 = 32n + 16$

$\rightarrow 3n^2 - 29n - 10 = 0$

$\rightarrow (3n+1)(n-10) = 0$

$n = -\frac{1}{3}$ $\qquad$ $n = 10$

n must be a positive integer → $n = 10$

(9) a) For $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ asymptotes are $\frac{x}{a} = \pm \frac{y}{b}$

when $x = 2$, $y = 0$

$\rightarrow \frac{4}{a^2} - 0 = 1 \rightarrow a^2 = 4 \rightarrow a = 2$

Asymptote $= y = 2x$ $\rightarrow$ $y = \frac{bx}{a}$ same as $y = 2x$

$\rightarrow \frac{b}{a} = 2 \rightarrow \frac{b}{2} = 2$

$\rightarrow b = 4$

b) Equation of line: $y - 0 = m(x-1)$

$\rightarrow y = mx - m$

Intercepts at:

$$\frac{x^2}{4} - \frac{(mx-m)^2}{16} = 1$$

$\rightarrow 4x^2 - (mx - m)^2 = 16$

$\rightarrow 4x^2 - m^2x^2 + 2m^2x - m^2 = 16$

$0 = m^2x^2 - 4x^2 - 2m^2x + m^2 + 16$

$\rightarrow (m^2-4)x^2 - (2m^2)x + (m^2+16) = 0$

c) For equal roots, $b^2 - 4ac = 0$

$\rightarrow (-2m^2)^2 - 4 \times (m^2-4)(m^2+16) = 0$

$\rightarrow 4m^4 - 4[m^4 + 12m^2 - 64] = 0$

$\rightarrow m^4 - m^4 - 12m^2 + 64 = 0$

$\rightarrow -12m^2 + 64 = 0$

$\rightarrow 12m^2 = 64 \rightarrow 3m^2 = 16$

d) $3m^2 = 16 \rightarrow m^2 = 16/3$

Find x-co-ordinate: $(m^2-4)x^2 - (2m^2)x + (m^2+16) = 0$

$\rightarrow (16/3 - 4)x^2 - 2(16/3)x + 16/3 + 16 = 0$

$\rightarrow 4/3\, x^2 - 32/3\, x + 16/3 + 16 = 0$

$(\times 3) \rightarrow 4x^2 - 32x + 64 = 0$

$\rightarrow x^2 - 8x + 16 = 0$

$(x-4)(x-4) = 0$

$\rightarrow x = 4$

Need to find y: $\quad x^2/2^2 - y^2/4^2 = 1$

$$\rightarrow 4^2/2^2 - y^2/4^2 = 1$$

$$\rightarrow 4 - y^2/16 = 1$$

$$64 - y^2 = 16$$

$$y^2 = 48$$

$$\rightarrow y = \pm\sqrt{48}$$

$$\rightarrow y = \pm 4\sqrt{3}$$

$(4, 4\sqrt{3})$ AND $(4, -4\sqrt{3})$